OCEN 689
Fall 2019

Homework 3, due Tue Oct 15

1. **Stability and accuracy:** Consider the ordinary differential equation

$$
\begin{aligned}
y' + (2 + 0.01t^2)y &= 0,\\
y(0) &= 4,
\end{aligned}
$$

Integrated from $t = 0$ to $t = 10$. The exact solution is

$$y(t) = 4e^{-2t-0.01t^3/3}.$$

(a) Solve this equation using the following numerical schemes: i) explicit (forwards) Euler; ii) implicit (backwards) Euler; iii) the trapezium rule. Use $h = 0.1, 0.2, 0.4, 1.0$. Plot all solutions on a graph and compute the L^2 of the error as a function of h.

(b) Discuss the stability and accuracy of each scheme.

(c) For each scheme, compute the maximum h for a stable solution (over the given domain) and discuss your estimate in terms of the results of part (a) (does your estimate agree with the simulation?).

2. **Stiff problems:** In this problem, we will compare the implicit Euler and trapezium rule methods for solving a stiff first order ODE. Consider the following differential equation:

$$\frac{dy}{dt} = \lambda(-y + \sin t), \quad y(0) = 0,$$

where $\lambda > 0$ is a real constant. The exact solution of the ODE is:

$$y = \left(\frac{\lambda^2}{1+\lambda^2}\right)\sin t + \left(\frac{\lambda}{1+\lambda^2}\right)(e^{-\lambda t} - \cos t).$$

Note that for large λ, the exact solution behaves like $\sin t$.

(a) Consider the time interval $[0, 10]$ and $\lambda = 5000$. Choose N steps, where $N = 100, 300, 500$. For each value of N, plot the numerical solutions and the exact solution on the same figure. What can you say about the accuracy of the numerical solutions? plot the L^2 norm as a function of N.

(b) Repeat part (a) with $y_0 = 0.1$. Is there a reason why you would prefer one numerical method over the other in this case?

3. **Heat equation stability:** We'll use our code from the Laplace equation (assignment 2) to model our first PDE. Consider the unsteady heat equation

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= D\frac{\partial^2 u(x,t)}{\partial x^2},\\ u(x,0) &= 2x(1-x),\\ u(0,t) &= 0,\\ u(1,t) &= 0,\end{aligned}$$

where $(x,t) \in [0,1] \times [0,1]$. We want to construct a numerical solution of the form

$$u_i^n = u(x_i, t_n), x_i = ih, i = 0, 1, \ldots, N, \quad t_n = n\delta t, n = 0, \ldots, M$$

(a) Set $D = 1$. In the interior, discretization in space gives for each u_i the ODE

$$\frac{du_i(t)}{dt} = \frac{D}{h^2}\left(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right)$$

This stencil must adjusted at the boundary to take into account the boundary conditions. Storing all unknowns $u_i, i = 1, \ldots, N-1$ into a vector $\vec{u}$, we can write all the ODEs together as the system of ODEs

$$\frac{d\vec{u}}{dt} = \frac{D}{h^2}A\vec{u},$$

where A is a tridiagonal matrix. Find A.

(b) Consider solving the equation using $h = 1/100, \delta t = \alpha h^2/2, \alpha = 0.5$. In MATLAB, create a fully discretized scheme using (i) implicit and (ii) explicit Euler. Your homework 2 code may be a nice start for your program. Are both solutions stable? For both schemes, plot your results for

$$t_n = 0, 0.01, 0.1, 0.2, 0.3, 0.4.$$

(c) Perform a convergence analysis in space: Compute the numerical solutions for $h = 1/5, 1/10, 1/20, 1/50, 1/100$. Fix the ratio $\delta t = \frac{h^2}{4}$ and take $h = 1/100$ to be your reference solution. For $t = 1$, plot the norm of the error as a function of h in logarithmic scale. What order of accuracy do you observe? Please plot two separate graphs, one for each scheme.

(d) For $h = 1/100$, take increasing values of $\alpha = 0.5, 0.6, \ldots, 1.1, 1.2$. At what point does explicit Euler become unstable? We'll get back to this stability limit in class in part III.

(e) For $h = 1/100$, take $\alpha = 10$. Is implicit Euler still stable? What is the order of accuracy?

(f) Now, for both implicit and explicit, set $D = -1, h = 1/100, \delta t = h^2/4$. Solve for $t = [0, 0.1]$. What happens? Explain why this behavior is expected.